# CENTER VORTEX MODEL FOR THE INFRARED SECTOR OF YANG-MILLS THEORY[a]

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A model for the infrared sector of $SU(2)$ Yang-Mills theory, based on magnetic vortices represented by (closed) random surfaces, is presented. The random surfaces, governed by an action penalizing curvature, are investigated using Monte Carlo methods on a hypercubic lattice. A low-temperature confining phase and a high-temperature deconfined phase are generated by this simple dynamics. After fixing the parameters of the model such as to reproduce the relation between deconfinement temperature and zero-temperature string tension found in lattice Yang-Mills theory, a surprisingly accurate prediction of the spatial string tension in the deconfined phase results. Furthermore, the Pontryagin index associated with the lattice random surfaces of the model is constructed. This allows to also predict the topological susceptibility; the result is compatible with measurements in lattice Yang-Mills theory. Thus, for the first time an effective model description of the infrared sector emerges which simultaneously and consistently reproduces both confinement and the topological aspects of Yang-Mills theory within a unified framework. Further details can be found in the e-prints hep-lat/9912003 and hep-lat/0004013, both to appear in Nucl. Phys. B.

## 1 Motivation

Diverse nonperturbative phenomena characterize strong interaction physics. Color charge is confined, chiral symmetry is spontaneously broken, and the axial $U(1)$ part of the flavor symmetry exhibits an anomaly. In principle, a theoretical tool exists which allows to calculate any observable associated with these phenomena, namely lattice gauge theory. Nevertheless, it is useful to concomitantly formulate effective models which concentrate on the relevant infrared degrees of freedom, and thus provide a clearer picture of the dominating physical mechanisms. This facilitates the exploration of problem areas which are difficult to access using the full (lattice) gauge theory.

Indeed, such model descriptions of the infrared sector exist. The most widely accepted picture of the confinement phenomenon at present is the dual

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superconductor mechanism,[1] based on Abelian monopole degrees of freedom. On the other hand, the $U_A(1)$ anomaly is most conveniently encoded in instanton degrees of freedom, which carry the relevant topological charge. Instanton models [2] are furthermore successful at modeling the spontaneous breaking of chiral symmetry. They, however, do not provide confinement, at least on the level of the simple model dynamics hitherto explored. Conversely, the connection of the aforementioned Abelian monopole degrees of freedom with the topological and chiral properties of the underlying theory, QCD, has not been completely clarified. Thus, the different nonperturbative phenomena have been explained on a separate footing, and no consistent, comprehensive picture of the infrared regime has emerged.

The vortex model presented in the following aims to bridge this gap. It simultaneously provides a quantitative description of both the confinement properties of $SU(2)$ Yang-Mills theory (including the finite-temperature transition to a deconfined phase), as well as the topological susceptibility, which encodes the $U_A(1)$ anomaly. This turns out to be possible on the basis of a simple effective model dynamics, detailed further below. At this stage, the chiral condensate, which represents the order parameter for the spontaneous breaking of chiral symmetry, has not been evaluated within the model. Some remarks on this next important point of investigation follow in section 5.

## 2 Chromomagnetic center vortices and their dynamics

Center vortices are closed lines of chromomagnetic flux in three-dimensional space; thus, they are described by closed two-dimensional world-surfaces in four-dimensional space-time. Their magnetic flux is quantized such that they contribute a phase corresponding to a nontrivial center element of the gauge group to any Wilson loop they are linked to (or, equivalently, whose minimal area they pierce). In the case of $SU(2)$ color discussed here, the only such nontrivial center phase is $(-1)$. For higher gauge groups, one must consider several possible fluxes carried by vortices.

As an illustrative example, consider a vortex surface located on the $x_1 = x_2 = 0$ plane, i.e. extending into the 3-4 direction. Such a vortex surface can be associated with a gauge field as follows.[c] Describe the 1-2 plane in polar coordinates, $(x_1, x_2) \rightarrow (r, \varphi)$, and set the $\varphi$-component of the gauge field to

$$A_\varphi = \frac{\sigma_3}{2r} \tag{1}$$

[c] It is possible to construct explicit gauge field representations for arbitrary vortex surface configurations.[3]

where $\sigma_3$ denotes the third Pauli matrix, encoding the color structure. All other components of the gauge field can be set to zero. Evaluating a circular Wilson loop of radius $R$ centered at $r = 0$ yields

$$W[R,0] = \frac{1}{2}\mathrm{Tr}\exp\left(i\int_0^{2\pi} d\varphi\, R\ \frac{\sigma_3}{2R}\right) = \frac{1}{2}\mathrm{Tr}\exp\left(i\pi\sigma_3\right) = -1 \qquad (2)$$

in accordance with the defining property of a vortex given above. Note that the result is independent of $R$; the flux of the vortex thus is localized at $r = 0$, and for any $R$, the full flux is encircled by the Wilson loop. This vortex is infinitely thin. The real physical vortex fluxes conjectured to describe the infrared aspects of the Yang-Mills ensemble within the vortex picture of course should be thought of as possessing a finite thickness;[d] however, for the purpose of evaluating deeply infrared observables, i.e. looking from far away, the thin idealization is adequate. Nevertheless, the thickness of the vortices will significantly influence the ansatz for the vortex dynamics presented further below, and the physical interpretation of that ansatz.

Having described the vortex by a gauge field, one can of course also associate a field strength with it. In view of the Lorentz structure of the field (1), the only nonvanishing tensor component of the field strength is $F_{r\varphi}$, i.e. the component associated with the two space-time directions perpendicular to the vortex. Lastly, it is important to note that an orientation can be associated with the vortex surface, corresponding to the two possible directions of the magnetic flux. Inverting the orientation simply means changing the sign of the gauge field, $A \rightarrow -A$. If one inserts the oppositely oriented vortex field in (2), one again obtains the value (-1) for the Wilson loop, i.e. the field $-A$ indeed also describes a vortex. Wilson loops, and thus the confinement properties of a vortex ensemble, are insensitive to the orientations of the vortices. However, since $A \rightarrow -A$ implies $F \rightarrow -F$, topological properties will in general turn out to be sensitive to the vortex orientations.

It should be noted that the chromomagnetic vortex fluxes described above correspond precisely to the flux domains found in the so-called Copenhagen vacuum.[5] Thus, the idea that these degrees of freedom may be relevant in the infrared sector of Yang-Mills theory is not new. What is new about the model to be discussed here is that the vortex picture has been developed into a quantitative tool, allowing to evaluate diverse physical obervables, that the vortex picture furthermore has been generalized to finite temperatures, including the deconfined phase, and, perhaps most importantly, that it has been extended to also subsume the topological properties of the Yang-Mills ensemble.

[d]Phenomenologically, such a physical thickness has e.g. been argued to be crucial for an explanation of the Casimir scaling behavior of adjoint representation Wilson loops.[4]

There are two complementary ways to think about vortex dynamics, i.e. to generate a vortex ensemble. On the one hand, one can devise an algorithm by which it is possible to identify and extract vortex structures from Yang-Mills field configurations.[6] Then, using the full Yang-Mills ensemble, as obtained e.g. in a standard lattice Monte Carlo experiment, in conjunction with the aforementioned algorithm, one can study the vortex physics induced by the full Yang-Mills dynamics. One drawback of such an approach is that no simplification is achieved compared with the standard lattice gauge calculation; rather, it allows a physical interpretation of the results of that calculation. This avenue will not be pursued here. Instead, vortices will be the only degrees of freedom entering the description from the very beginning, and a simple effective model dynamics for the vortices will be assumed, with the aim of ascertaining how far such a simple model will carry.

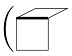
Specifically, to arrive at a tractable model, the vortex world-surfaces will be composed of elementary squares (plaquettes) on a hypercubic lattice. The spacing of this lattice will be a fixed physical quantity related to the thickness of the vortex fluxes already mentioned further above; the lattice prevents an arbitrarily close packing of the vortices. Thus, it is not envisaged to eventually take the lattice spacing to zero, and accordingly renormalize the coupling constants, such as to arrive at a continuum theory. Rather, the lattice spacing represents the fixed physical cutoff one expects to be present in any infrared effective theory, and thus also delineates the ultraviolet limit of validity of the model. If one wants to refine the model such as to eliminate the artificial hypercubic nature of the vortex surfaces, one has to replace the lattice spacing by some other ultraviolet cutoff. For instance, if the surfaces are represented as triangulations, a minimal area of the elementary triangles could take on this role.

On the hypercubic lattice adopted here, the vortex surfaces will be regarded as random surfaces. This is not such a far-fetched idea; heuristically, the vortex degrees of freedom are magnetic degrees of freedom, dual to the electric degrees of freedom in terms of which one initially defines gauge theories.[e] Thus, in the infrared regime, where the usual electric degrees of freedom become strongly coupled, one would conversely expect magnetic vortex degrees of freedom to be weakly coupled, i.e. weakly correlated. The reason the magnetic degrees of freedom are nevertheless organized into vortex lines lies in the constraint of continuity for the magnetic flux (which represents the dual to the usual Gauß law constraint).

The random surfaces describing the vortices will be generated by Monte

[e]This is particularly manifest in the fact that the lattice on which center vortices are defined is dual to the one on which standard lattice gauge theory is constructed.

Carlo methods; the weight function specifying the ensemble depends on the curvature of the surfaces as follows. Every instance of a link on the lattice being common to two plaquettes which are part of a vortex surface, but which do not lie in the same plane, is penalized by an action increment $c$. Thus the action can be represented pictorially as

$S_{curv} = c \cdot \# \Big($  $\Big)$ (3)

Note that several such pairs of plaquettes can occur for any given link. For instance, if all six plaquettes attached to a given link are part of a vortex surface, this implies a contribution $12c$ to the action.

Since this construction seems rather ad hoc, a comment is in order at this point. As the random surface model discussed here is meant to represent an infrared effective theory, it seems natural to expand the action in a gradient expansion. In such an expansion, one would expect a Nambu-Goto term as the leading term, i.e. a contribution to the action proportional to the total surface area. Only in the next to leading order, curvature terms appear. Such a generalized model has indeed been investigated,[7] containing both a Nambu-Goto action as well as the curvature term $S_{curv}$ given above. It turns out that, to a large extent, one can trade the effects of the two action terms off against one another; more precisely, in the plane of coupling coefficients corresponding to the two terms, one finds lines on which confinement physics (specifically all string tensions, as functions of temperature), as discussed in the next section, is invariant up to ten percent effects. This is not entirely surprising, since surfaces with a large area typically will also contain a large amount of curvature, and vice versa (up to exceptions which are entropically suppressed). Thus, both coupling coefficients to some extent have similar effects. The aforementioned lines of approximately invariant physics happen to cross the axis on which the Nambu-Goto coefficient vanishes (but not the axis on which the curvature coefficient $c$ does). Thus, one can restrict oneself to the models given by the action (3) without loss of generality.

As a last remark, for the purpose of studying topological properties, cf. section 4, it is necessary to also specify orientations for the vortex surfaces (this was already hinted at above). Section 4 contains more details on this point.

## 3 Confinement and Deconfinement

Given the random surface dynamics defined in the previous section, it is now straightforward to evaluate Wilson loops, using the fundamental property of vortices that they modify any Wilson loop by a phase factor $(-1)$ whenever

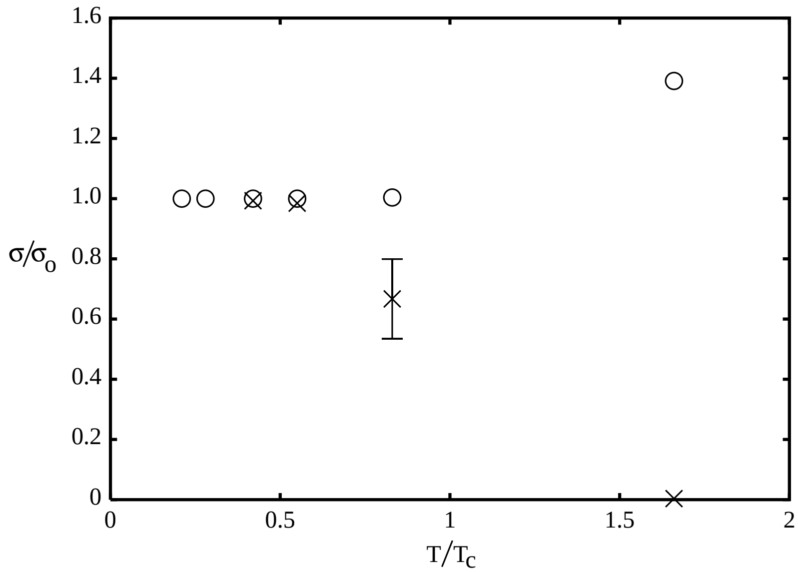



Figure 1: String tension between static color sources (crosses) and spatial string tension (circles) as a function of temperature.

they pierce its minimal area. Such measurements can furthermore be carried out at several temperatures, by adjusting the extension of (Euclidean) space-time in the time direction; this extension is identified with the inverse temperature of the ensemble. At finite temperatures, static quark potentials are given by Polyakov loop correlators; their properties in the presence of vortex fluxes are completely analogous to the properties of Wilson loops. Qualitatively, as long as the curvature coefficient $c$ is not too large, one finds a confined phase (non-zero string tension) at low temperatures, and a phase transition to a high-temperature deconfined phase. In order to make the correspondence to full $SU(2)$ Yang-Mills theory quantitative, one can adjust $c$ such as to reproduce the ratio of the deconfinement temperature to the square root of the zero-temperature string tension, $T_C/\sqrt{\sigma_0} = 0.69$. This happens at the value $c = 0.24$. Furthermore, by setting $\sigma_0 = (440\,\mathrm{MeV})^2$ to fix the scale, one extracts from the measurement of $\sigma_0 a^2$ the lattice spacing $a = 0.39\,\mathrm{fm}$. As discussed in the previous section, this is a fixed physical quantity, related to the thickness of the vortices, which represents the ultraviolet limit of validity of the effective vortex model.

Fig. 1 displays the result of string tension measurements on $16^3 \times N_t$

lattices, as a function of temperature, for the choice of curvature coefficient $c = 0.24$. Whereas the quantitative behavior of the static quark string tension, represented by the crosses in Fig. 1, has been fitted using the freedom in the choice of $c$, the so-called spatial string tension $\sigma_s$, represented by the circles in Fig. 1, can now be predicted. In the high-temperature regime, it begins to rise with temperature; the value obtained at $T = 1.67\,T_C$, namely $\sigma_s(T = 1.67\,T_C) = 1.39\sigma_0$, corresponds to within 1% with the value measured in full $SU(2)$ Yang-Mills theory.[8]

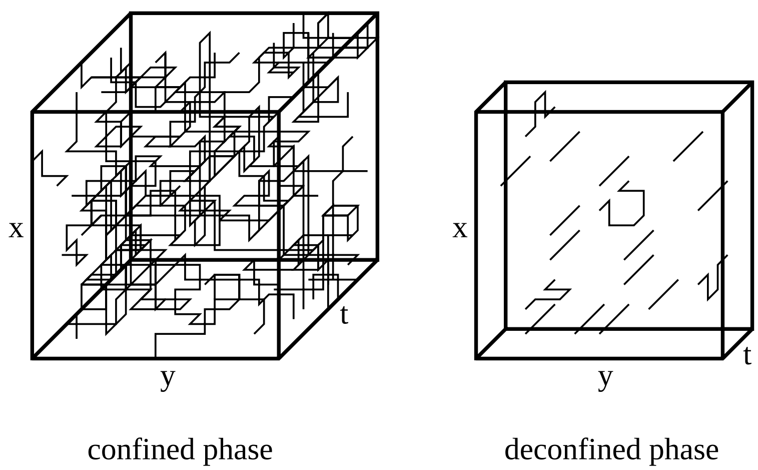



Figure 2: Typical vortex configurations in the confined and the deconfined phases.

The confined and deconfined phases can alternatively be characterized by the percolation properties[7] of the vortex clusters in space slices of the lattice universe, i.e. slices defined by keeping one space coordinate fixed. In such a slice of space-time, vortices are represented by closed lines, cf. Fig. 2. In the confined phase, these lines percolate throughout (sliced) space-time, whereas in the deconfined phase, they form small, isolated clusters, which, more specifically, wind around the universe in the Euclidean time direction (and are closed by virtue of the periodic boundary conditions). Also, simple heuristic arguments can be given[7] which explain why confinement should be associated with vortex percolation. The percolation characteristics of the surfaces in the vortex model precisely mirror the ones found for vortex structures extracted from full lattice Yang-Mills configurations.[9]

## 4 Topology

Next to the confinement phenomenon, the topological properties of the Yang-Mills ensemble constitute an important nonperturbative aspect of strong interaction theory, as already highlighted in section 1. These properties are encoded in the topological winding number

$$Q = \frac{1}{32\pi^2} \int d^4x \, \epsilon_{\mu\nu\lambda\tau} \ \mathrm{Tr} \ F_{\mu\nu} F_{\lambda\tau} \ . \tag{4}$$

In view of this expression, to generate a nonvanishing topological density at a certain point in space-time, the field strength there must have nonvanishing tensor components such that the corresponding Lorentz indices span all four space-time directions. For example, a nonvanishing $F_{12}$ in conjunction with a nonvanishing $F_{34}$ would fulfil this requirement. In order to generate nonvanishing $F_{12}$, a vortex surface segment (locally) running in 3-4 direction is necessary, as discussed in section 2; conversely, to generate nonvanishing $F_{34}$, one needs a surface segment running in 1-2 direction. Thus, a nontrivial contribution to the topological winding number is e.g. generated by a self-intersection point of the vortex surfaces, where a segment running in 3-4 direction intersects a segment running in 1-2 direction. Quantitatively,[3] the contribution specifically of such a self-intersection point has modulus 1/2. In general, all singular points of a surface configuration contribute, where a singular point is defined as a point at which the set of tangent vectors to the surface configuration spans all four space-time directions. Self-intersection points are but the simplest example of such singular points; there also exist writhing points, at which the surface is twisted in such a way as to generate a singular point in the above sense. These writhing points actually turn out to be statistically far more important than the self-intersection points in the random surface ensemble discussed here.[10]

As already indicated in section 2, the orientation of the vortex surfaces enters the topological winding number, since it is encoded in the sign of the field strength. This should be contrasted with the Wilson loop, which is insensitive to the orientation. The random surfaces of the vortex ensemble should not be considered to be globally oriented (in fact, most configurations are not even globally orientable); they in general should be composed of patches of alternating orientation. Given a vortex surface from the ensemble defined by (3), it is straightforward to randomly assign orientations to the plaquettes making up the surface. By biasing this procedure with respect to the relative orientation of neighboring plaquettes, different mean sizes of the oriented patches making up the surface can be generated; equivalently, the density of patch boundary lines can be adjusted. This density strictly speaking constitutes an

additional parameter of the model, which cannot be fixed using the confinement properties due to the fact that the Wilson loop is insensitive to the vortex orientation. A priori, one might expect e.g. the topological susceptibility, the measurement of which is presented below, to depend on this parameter. This would mean that the topological susceptibility can possibly be fitted, but not predicted. In actual fact, it turns out that this quantity is independent of the aforementioned density within the error bars. The reasons for this can be understood in detail in terms of the properties of the random surface ensemble.[10] The measurement of the topological susceptibility exhibited below therefore does represent a genuine quantitative prediction of the vortex model.

At first sight, given the surfaces, including their orientation, it may seem straightforward to locate their singular points and thus evaluate the topological winding number. However, surfaces constructed on a hypercubic lattice, as done in the present formulation of the vortex model, contain ambiguities which would not appear if one were dealing with arbitrary surfaces in continuous space-time. For one, the lattice surfaces in general self-intersect not at points, but along whole intersection lines; in an ensemble of arbitrary continuum surfaces, such configurations would represent a set of measure zero. Likewise, the boundaries of the oriented patches making up the surfaces discussed above coincide with singular points of the surfaces with a finite probability; again, such configurations would represent a set of measure zero in an ensemble of arbitrary continuum surfaces. These ambiguities are resolved by interpreting the surfaces on the hypercubic lattice as coarse-grained representations of other surfaces which slightly deviate from the former ones. Physically, this is consistent; in view of the thickness of the vortices, the location of the associated surfaces is only defined up to arbitrary deviations shorter than this thickness. Thus, in practice, the initial vortex surfaces can be transferred onto a finer lattice, and arbitrary short-range deformations can be carried out on the surfaces such as to eliminate all the ambiguities highlighted above. Then the topological winding number can indeed be extracted unambiguously.

Note therefore that the calculation of the topological properties within the vortex model on a hypercubic lattice is afflicted with difficulties which are reminiscent of the ones occuring in standard lattice gauge theory. The elimination of ambiguities discussed above is analogous to an "inverse blocking" prescription for Yang-Mills link configurations.

Having thus obtained an algorithm allowing to evaluate the topological winding number $Q$ of vortex surface configurations on a hypercubic lattice, it is possible to measure the topological susceptibility $\chi = \langle Q^2 \rangle / V$ of the vortex ensemble, where $V$ denotes the space-time volume under consideration. The result is exhibited in Fig. 3 as a function of temperature.

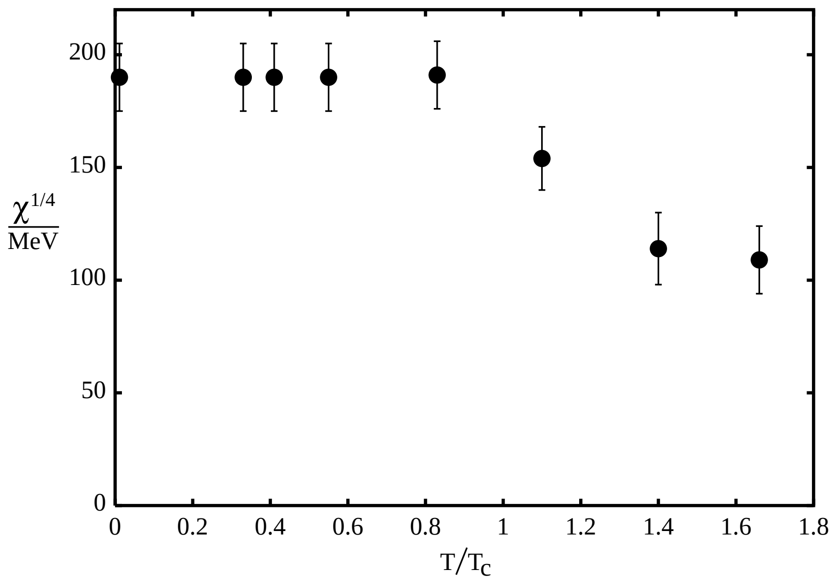



Figure 3: Fourth root of the topological susceptibility as a function of temperature.

Quantitatively, this result is compatible with measurements in full Yang-Mills theory.[11] The vortex model thus provides, within one common framework, a simultaneous, consistent description of both the confinement properties as well as the topological properties of the $SU(2)$ Yang-Mills ensemble. It should remarked that, also as far as the vortex structures extracted from lattice Yang-Mills configurations are concerned, evidence has been obtained that these structures encode the topological characteristics of the gauge fields: Removing all vortices from a lattice gauge configuration has the effect of transferring the configuration to the topologically trivial ($Q = 0$) sector.[12]

## 5 Outlook

One obvious generalization of the present work is the treatment of $SU(3)$ color. The vortex model appropriate for this color group will in some respects exhibit qualitatively different behavior. Since there are two nontrivial center elements in the $SU(3)$ group, namely the phases $e^{\pm i2\pi/3}$, one will have to distinguish two distinct vortex fluxes. The main qualitative difference in the topology of

vortex configurations will be the presence of vortex branchings. In the $SU(3)$ theory, there are branchings which respect flux conservation; a vortex carrying one type of vortex flux can split into two vortices carrying the other type of vortex flux. This difference in the topological character of the configurations is expected to lead to a change in the order of the deconfinement phase transition from second order for $SU(2)$ to first order for $SU(3)$, as observed in full Yang-Mills lattice experiments.[13]

To render the vortex model a useful quantitative tool, it is furthermore important to couple the vortices to quark degrees of freedom. Foremost, it must be ascertained whether the vortex ensemble induces the spontaneous breaking of chiral symmetry to a sufficient degree; i.e., the chiral condensate must be evaluated quantitatively. Technically, this implies constructing the Dirac operator in a vortex background. In this respect, the vortex picture has an important advantage to offer. As hinted in section 2, any arbitrary vortex surface can be associated with a continuum gauge field; this trivially includes surfaces which happen to be made up of elementary squares, e.g. plaquettes on a hypercubic lattice. Therefore, the Dirac operator can be constructed directly in the continuum, and some of the difficulties associated with lattice Dirac operators, such as fermion species doubling, may be avoidable. From a physical point of view, it is tempting to speculate that a model with the correct topological properties should also correctly account for the chiral condensate, in view of the corresponding success of instanton models [2] in describing the spontaneous breaking of chiral symmetry.

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